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# On some group properties of Newtonian static star structure equations

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**Abstract.** By using the Lie group theory, symmetries of the system of equations, describing Newtonian static stars in radiative equilibrium, are investigated. It turns out that the most general symmetries are those resulting from quasi-homologous transformations. These symmetries enforce a corresponding equation of state. Stromgren's homologous stars are a special case of this, more general, class of solutions.

## 1. Introduction

One of the main problems in the group analysis of differential equations is the investigation of properties of the group admissible by the differential equation structure. In the set of all solutions the action of an admissible group induces a certain algebraic structure which can be used to find a family of new solutions from the known ones (Ovsiannikov 1982).

In the present work we apply Lie group theory to investigate group properties of the system of four structure equations describing Newtonian static stars in radiative equilibrium. Two of these equations, namely the hydrostatic equilibrium equation and the mass continuity equation, were investigated by Collins from the group theory point of view (Collins 1977).

As is well known from Stromgren's theorem (Stromgren 1936) new solutions, for such a system of equations, can be obtained from the known ones through the homologous transformations. New solutions will describe new configurations with different mass, radius and chemical composition (the so-called homologous stars). We generalise this result by introducing the notion of quasi-homologous stars, i.e. the stars whose equation of state admits quasi-homology symmetries. Homologous stars are a special case of quasi-homologous ones. At the present stage of our investigation, this should be treated as a purely mathematical result, although it cannot be excluded that the obtained dependencies between luminosity and temperature, mass and temperature and so on could be employed in a manner similar to that done by Stromgren (1936) to fully interpret the Hertzsprung–Russell diagram.

The material is organised as follows. Section 2 gives the necessary rudiments of the group method to analyse symmetries of differential equations (we closely follow Collins' presentation). The quasi-homologous structure of the Newtonian star equations is analysed in § 3. In § 4 we investigate equations of state enforced by homology symmetry postulates. Finally the main results of our research are summarised.

## 2. Mathematical background

In the present work we consider differential equation systems of the following form

$$\mathrm{d}u^i/\mathrm{d}x = f^i(x, u^1, \ldots, u^m) \qquad i = 1, \ldots, m. \tag{1}$$

We consider point transformations generated by the infinitesimal operator

$$X = \xi(x, u^1, \ldots, u^m)\frac{\partial}{\partial x} + \sum_{i=1}^{m} \eta^i(x, u^1, \ldots, u^m)\frac{\partial}{\partial u^i}. \tag{2}$$

For the infinitesimal operator $X$ there exist $m$ independent invariants which are solutions to the following system:

$$\frac{\mathrm{d}x}{\xi(x, u^1, \ldots, u^m)} = \frac{\mathrm{d}u^1}{\eta^1(x, u^1, \ldots, u^m)} = \cdots = \frac{\mathrm{d}u^m}{\eta^m(x, u^1, \ldots, u^m)}. \tag{3}$$

The point transformation generated by $X$ is called homologous if $\xi = ax$ and $\eta^i = g^i u^i$, where $a$, $g^i$, $i = 1, \ldots, m$ are constants. It can easily be seen that system (1) is similarity invariant in this case.

A natural extension of this special case leads us to the notion of quasi-homologous transformations: $\xi = \xi(x)$, $\eta^i = \eta^i(u^i)$, $i = 1, \ldots, m$. System (1) admits the infinitesimal operator (2) if and only if the following condition is satisfied:

$$\frac{\partial \eta^i}{\partial x} + \sum_{j=1}^{m}\left(\frac{\mathrm{d}u^j}{\mathrm{d}x}\frac{\partial \eta^i}{\partial u^j} - \frac{\mathrm{d}u^i}{\mathrm{d}x}\frac{\mathrm{d}u^j}{\mathrm{d}x}\frac{\partial \xi}{\partial u^j}\right) - \frac{\mathrm{d}u^i}{\mathrm{d}x}\frac{\partial \xi}{\partial x} - X(f^i) = 0 \qquad i = 1, \ldots, m. \tag{4}$$

Condition (4) tells us whether the symmetry operator $X$ is admitted by system (1) or not (Ovsiannikov 1982). It is easily seen that, in the case of a quasi-homologous transformation, equation (4) assumes the form

$$\frac{\mathrm{d}\eta^i}{\mathrm{d}u^i} - \frac{\mathrm{d}\xi}{\mathrm{d}x} = X(\ln f^i) \qquad i = 1, \ldots, m. \tag{5}$$

## 3. Quasi-homologous transformations of structure equations

The structure equations for a Newtonian static star are the following (Schwarzschild 1958):

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{GM\rho}{r^2} \qquad \text{hydrostatic equilibrium} \tag{6}$$

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \rho \qquad \text{mass continuity} \tag{7}$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \varepsilon(\rho, T) \qquad \text{thermal equilibrium} \tag{8}$$

either

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi ac}\frac{\rho}{r^2}\frac{L}{T^3}\varkappa(\rho, T) \qquad \text{radiative equilibrium} \tag{9}$$

or

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{\mathrm{d}p}{\mathrm{d}r} \qquad \text{adiabatic convective equilibrium} \tag{10}$$

where $M$ is the mass within the sphere of radius $r$, $\rho$ the density, $p$ the pressure, $L$ the luminosity at the surface of the sphere of radius $r$, $T$ the temperature, $\varepsilon$ the energy generation rate, $\varkappa$ the opacity, $G$ the gravitational constant, $c$ the velocity of light and $a$ the Stefan-Boltzmann constant.

First we shall consider the case of radiative equilibrium. Assuming the equation of state in the form $p=p(\rho, T)$ we can rewrite (6) in the more convenient form

$$\frac{\mathrm{d}\rho}{\mathrm{d}r}=\left(\frac{\partial p}{\partial \rho}\right)^{-1}\left(-GM+\frac{3}{16\pi ac}\frac{L}{T^3}\varkappa\frac{\partial p}{\partial T}\right)\frac{\rho}{r^2}. \tag{11}$$

Now, we look for the symmetry transformations of equations (11), (7), (8) and (9) generated by the operator

$$X=\xi(r)\frac{\partial}{\partial r}+\eta^1(\rho)\frac{\partial}{\partial \rho}+\eta^2(M)\frac{\partial}{\partial M}+\eta^3(L)\frac{\partial}{\partial L}+\eta^4(T)\frac{\partial}{\partial T}. \tag{12}$$

If we denote

$$f=-GM+\frac{3}{16\pi ac}\frac{L}{T^3}\varkappa\frac{\partial p}{\partial T}$$

the admissible equations take the form

$$\begin{aligned}\frac{\mathrm{d}\eta^1}{\mathrm{d}\rho}-\frac{\mathrm{d}\xi}{\mathrm{d}r}=&-\frac{2\xi}{r}+\frac{\eta^1}{\rho}-\eta^1\frac{\partial}{\partial\rho}\left(\ln\frac{\partial p}{\partial\rho}\right)+\frac{\eta^1}{f}\left[\frac{3}{16\pi ac}\frac{L}{T^3}\frac{\partial}{\partial\rho}\left(\varkappa\frac{\partial p}{\partial T}\right)\right]-\frac{\eta^2 G}{f}\\&+\frac{\eta^3}{f}\left(\frac{3}{16\pi ac}\frac{\varkappa}{T^3}\frac{\partial p}{\partial T}\right)+\eta^4\left[-\frac{\partial}{\partial T}\left(\ln\frac{\partial p}{\partial\rho}\right)+\frac{1}{f}\frac{3L}{16\pi ac}\frac{\partial}{\partial T}\left(\frac{\varkappa}{T^3}\frac{\partial p}{\partial\rho}\right)\right]\end{aligned} \tag{13}$$

$$\frac{\mathrm{d}\eta^2}{\mathrm{d}M}-\frac{\mathrm{d}\xi}{\mathrm{d}r}==\frac{2\xi}{r}+\frac{\eta^1}{\rho} \tag{14}$$

$$\frac{\mathrm{d}\eta^3}{\mathrm{d}L}-\frac{\mathrm{d}\xi}{\mathrm{d}r}=\frac{2\xi}{r}+\frac{\eta^1}{\rho}+\frac{\eta^1}{\varepsilon}\frac{\partial\varepsilon}{\partial\rho}+\frac{\eta^4}{\varepsilon}\frac{\partial\varepsilon}{\partial T} \tag{15}$$

$$\frac{\mathrm{d}\eta^4}{\mathrm{d}T}-\frac{\mathrm{d}\xi}{\mathrm{d}r}=-\frac{2\xi}{r}+\frac{\eta^1}{\rho}+\frac{\eta^1}{\varkappa}\frac{\partial\varkappa}{\partial\rho}+\frac{\eta^3}{L}+\eta^4\left(-\frac{3}{T}+\frac{1}{\varkappa}\frac{\partial\varkappa}{\partial T}\right). \tag{16}$$

Since the right-hand side of equation (14) depends only on $\rho$, therefore $\eta^1=\alpha_1\rho$ where $\alpha_1$ is a constant. It is easy to verify, by the same argument and by substitution into equations (13)-(16), that also

$$\eta^2=\alpha_2 M \qquad \eta^3=\alpha_3 L \qquad \xi=\tfrac{1}{3}(\alpha_2-\alpha_1)r$$

where $\alpha_2$, $\alpha_3$ are constants. By substituting these equations into the system (13)-(16) one obtains

$$\left(\frac{\mathrm{d}\eta^4}{\mathrm{d}T}-\frac{2}{3}\alpha_2-\frac{4}{3}\alpha_1\right)\frac{\partial p}{\partial T}+\alpha_1\rho\frac{\partial^2 p}{\partial\rho\partial T}+\eta^4\frac{\partial^2 p}{\partial T^2}=0 \tag{17}$$

$$-\tfrac{2}{3}\alpha_2-\tfrac{1}{3}\alpha_1+\alpha_1\rho\frac{\partial}{\partial\rho}\left(\ln\frac{\partial p}{\partial\rho}\right)+\eta^4\frac{\partial}{\partial T}\left(\ln\frac{\partial p}{\partial\rho}\right)=0 \tag{18}$$

$$\tfrac{4}{3}\alpha_1-\tfrac{1}{3}\alpha_2+\alpha_3-\frac{\mathrm{d}\eta^4}{\mathrm{d}T}+\alpha_1\frac{\rho}{\varkappa}\frac{\partial\varkappa}{\partial\rho}+\eta^4\left(-\frac{3}{T}+\frac{1}{\varkappa}\frac{\partial\varkappa}{\partial T}\right)=0 \tag{19}$$

$$(\alpha_3-\alpha_2)\varepsilon=\alpha_1\rho\frac{\partial\varepsilon}{\partial\rho}+\eta^4\frac{\partial\varepsilon}{\partial T}. \tag{20}$$

Equations (19) and (20) imply that the opacity coefficient $\varkappa$ and the energy generation rate $\varepsilon$ are determined by the property of quasi-homologous temperature transformation generated by the component $\eta^4(T)\partial/\partial T$. This gives us the following solutions:

$$\varphi\left\{\rho \exp\left(-\alpha_1 \int_{T_0}^{T} \frac{\mathrm{d}t}{\eta^u(t)}\right), \varkappa \exp\left[-\int_{T_0}^{T}\left(\frac{-\mathrm{d}\eta^4/\mathrm{d}t - 3\eta^4/t - \alpha_2/2 + \frac{4}{3}\alpha_1}{\eta^4(t)}\right)\mathrm{d}t\right]\right\} = 0 \tag{21}$$

$$\psi\left(\rho \exp\left(-\alpha_1 \int_{T_0}^{T} \frac{\mathrm{d}t}{\eta^4(t)}\right), \rho\, \varepsilon^{\alpha_1/(\alpha_3-\alpha_2)y}\right) = 0 \tag{22}$$

where $\varphi$, $\psi$ are arbitrary functions.

There still remain equations (17) and (18) to be solved. Since we have some freedom in choosing the function $\eta^4(T)$, and the function $p(\rho, T)$ is never precisely known, we shall look for those equations of state $p = p(\rho, T)$ that are enforced by the transformations generated by operator (2). It is easy to check, by using equations (17) and (18), that $p = p(\rho, T)$ satisfies the continuity condition $\partial^2 p/\partial\rho\partial T = \partial^2 p/\partial T\partial\rho$ which in fact is a consistency condition for (17) and (18). Depending on the hyperbolic or parabolic character of equations (17) and (18) four cases can be distinguished (see table 1). The solutions for cases I–IV are the following.

*Case I*

$$X = \frac{\alpha_2 - \alpha_1}{3} r\frac{\partial}{\partial r} + \alpha_1 \rho\frac{\partial}{\partial \rho} + \alpha_2 M\frac{\partial}{\partial M} + \alpha_3 L\frac{\partial}{\partial L} + \eta^4(T)\frac{\partial}{\partial T}.$$

The standard method of reduction to the canonical form, when applied to equation (18), gives

$$\frac{\partial^2 p(x, y)}{\partial x\partial y} = \frac{\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2}{\eta^4(y)}\frac{\partial p}{\partial x} \tag{23}$$

where

$$x = \rho \exp\left(-\int_{T_0}^{T} \frac{\alpha_1\, \mathrm{d}t}{\eta^4(t)}\right) \qquad y = T.$$

In this case, the general solution of equation (23) assumes the form (Vladimirov 1974):

$$p(x, y) = h(y) + \int_{x_0}^{x} g(\zeta)\,\mathrm{d}\zeta \exp\left(\int_{y_0}^{y} A(t)\,\mathrm{d}t\right) \tag{24}$$

where

$$A(t) = \frac{\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2}{\eta^4(t)}.$$

$h(y)$ is of $C^2$ class and $g(\zeta)$ is of $C^1$ class of differentiability.

**Table 1.**

| | $\alpha_1 \neq 0$ | $\alpha_1 = 0$ |
|---|---|---|
| $\eta^4(T) \neq 0$ | I. (17) hyperbolic<br>(18) hyperbolic | III. (17) parabolic<br>(18) hyperbolic |
| $\eta^4(T) \equiv 0$ | II. (17) hyperbolic<br>(18) parabolic | IV. (17) parabolic<br>(18) parabolic |

By substituting the general solution (24) in equation (17) we obtain the additional condition for $h(y)$

$$\eta^4(y)h''(y)+\left(\tfrac{4}{3}\alpha_1+\tfrac{2}{3}\alpha_2-\frac{\mathrm{d}\eta^4}{\mathrm{d}y}\right)h'(y)=0. \tag{25}$$

The solution of this equation is

$$h(y)=C_1\int_{y_0}^{y}\exp\left(\int_{\tau_0}^{\tau}(\tfrac{4}{3}\alpha_1+\tfrac{2}{3}\alpha_2)\frac{\mathrm{d}t}{\eta^4(t)}\right)\frac{\mathrm{d}\tau}{\eta^4(\tau)}+C_2 \tag{26}$$

where $C_1$ and $C_2$ are constants.

This solution implies that in equation (24) there is still the freedom in choosing a function $g=g(\zeta)$.

*Case II*

$$X=\frac{\alpha_2-\alpha_1}{3}r\frac{\partial}{\partial r}+\alpha_2M\frac{\partial}{\partial M}+\alpha_3L\frac{\partial}{\partial L}+\eta^4(T)\frac{\partial}{\partial T}.$$

By proceeding in the same way as in the previous case we obtain

$$p(\rho,T)=h(T)+\int_{\rho_0}^{\rho}g(\rho')\exp\left(\int_{T_0}^{T}\frac{2\alpha_2\mathrm{d}t}{3\eta^4(t)}\right)\mathrm{d}\rho' \tag{27}$$

where

$$h(T)=C_1\int_{T_0}^{T}\exp\left(\tfrac{2}{3}\alpha_2\int_{t_0}^{t}\frac{\mathrm{d}\tau}{\eta^4(\tau)}\right)\frac{\mathrm{d}t}{\eta^4(t)}+C_2$$

and $h, g$ satisfy differentiability conditions as in the previous case and $C_1, C_2$ are constants.

*Case III*

$$X=\frac{\alpha_3-\alpha_2}{3}r\frac{\partial}{\partial r}+\alpha_1\rho\frac{\partial}{\partial\rho}+\alpha_2M\frac{\partial}{\partial M}+\alpha_3L\frac{\partial}{\partial L}.$$

Similar calculations give

$$p(\rho,T)=h(\rho)+\int_{T_0}^{T}g(t)\,\mathrm{d}t\rho^{\frac{4}{3}+\frac{2}{3}\alpha_2/\alpha_1} \tag{28}$$

where

$$h(\rho)=C_1\rho^{\frac{4}{3}+\frac{2}{3}\alpha_2/\alpha_1}+C_2$$

$h, g$ satisfy the differentiability conditions as previously and $C_1, C_2$ are constants.

*Case IV*

$$X=\frac{\alpha_2-\alpha_1}{3}r\frac{\partial}{\partial r}+\alpha_2M\frac{\partial}{\partial M}+\alpha_3L\frac{\partial}{\partial L}.$$

In this case we have the following solutions:

$$p=\text{constant} \tag{29}$$

$$p=p(\rho)\qquad\text{for }\tfrac{2}{3}\alpha_2+\tfrac{1}{3}\alpha_1=0 \tag{30}$$

$$p=p(T)\qquad\text{for }\tfrac{2}{3}\alpha_2+\tfrac{4}{3}\alpha_1=0. \tag{31}$$

We can construct finite transformations and group invariants. In case I, for example, we obtain four independent invariants $L_0$, $M_0$, $r_0$ and $\rho_0$:

$$\begin{aligned} L_0 &= L \exp\left(-\alpha_3 \int_{T_0}^{T} \frac{\mathrm{d}t}{\eta^4(t)}\right) \\ M_0 &= M \exp\left(-\alpha_2 \int_{T_0}^{T} \frac{\mathrm{d}t}{\eta^4(T)}\right) \\ r_0 &= r \exp\left(\frac{\alpha_1-\alpha_2}{3} \int_{T_0}^{T} \frac{\mathrm{d}t}{\eta^4(t)}\right) \\ \rho_0 &= \rho \exp\left(-\alpha_1 \int_{T_0}^{T} \frac{\mathrm{d}t}{\eta^4(t)}\right). \end{aligned} \tag{32}$$

By using these invariants, we can construct the new families of solutions, for instance

(i) if $L(T)$ is the solution of (11), (7), (8) and (9), then also

$$L(E(T)) \exp\left(-\alpha_3 \int_{E(T_0)}^{E(T)} \frac{\mathrm{d}t}{\eta^4(t)}\right)$$

is the solution, where $E(T)$ is a finite transformation of $T$,

(ii) if $M(T)$ is the solution of (11), (7), (8) and (9), then also

$$M(E(T)) \exp\left(-\alpha_2 \int_{E(T_0)}^{E(T)} \frac{\mathrm{d}t}{\eta^4(t)}\right)$$

is the solution.

## 4. Homologous symmetry transformations of structure equations

It is well known that equations (11), (7), (8) and (9) admit similarity symmetries for the state equation of an ideal gas: $p \sim \rho T$. This fact induces a certain class of homologous solutions to the system. Some classical results have been obtained by Stromgren (1936).

In order to investigate how general our results are, let us assume the rescaling symmetry: $\eta^4(T) = \alpha_4 T$. Then, in case I, for instance, the following equation of state is enforced:

$$p(x, T) = \frac{C_1}{\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2 + \alpha_4} T^{(4\alpha_1+2\alpha_2)/3\alpha_4} + \int_{x_0}^{x} g(\zeta)\,\mathrm{d}\zeta \frac{T^{1+(4\alpha_1+2\alpha_2)/3\alpha_4}}{\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2 + \alpha_4}. \tag{33}$$

One should notice that equation (33) contains the following form of the equation of state:

$$p = a\rho^A + bT^B + c\rho^C T^D. \tag{34}$$

The infinitesimal operator corresponding to the homologous transformations takes the form

$$X = \frac{\alpha_2 - \alpha_1}{3} r\frac{\partial}{\partial r} + \alpha_1 \rho \frac{\partial}{\partial \rho} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L} + \alpha_4 T \frac{\partial}{\partial T}. \tag{35}$$

The operator given by equation (35) has four independent invariants, for instance

$$J_1 = \rho r^{3\alpha_1/(\alpha_1-\alpha_2)} \qquad J_2 = \rho M^{-\alpha_1/\alpha_2}$$
$$J_3 = LM^{-\alpha_3/\alpha_2} \qquad J_4 = LT^{-\alpha_3/\alpha_4}.$$

By using these invariants we can arrive at various homology theorems, for example the theorem associated with $J_1$: if $\rho(r)$ is the solution of (11), (7), (8) and (9), then $\rho\{r\exp[(\alpha_2-\alpha_1)/3]\}\,e^{-\alpha_1}$ is also the solution.

From $J_3$ we obtain $L\sim M^{\alpha_3/\alpha_2}$ which corresponds to the well known Eddington mass-luminosity dependence. The infinitesimal operator (35) generates a Lie algebra spanned by the basis operators

$$X_1=-\tfrac{1}{3}r\partial/\partial r+\rho\partial/\partial\rho$$
$$X_2=\tfrac{1}{3}r\,\partial/\partial r+M\,\partial/\partial M$$
$$X_3=L\,\partial/\partial L$$
$$X_4=T\,\partial/\partial T.$$

Now we can construct solutions associated with these operators. Let us start with $X_1$. Since $\rho r^3$ is an invariant we can have $\rho(r)=\rho_0 r^{-3}$. By substituting this in equation (7) and by integrating it over $r$, we obtain $M(r)=M_0+4\pi\rho_0\ln r$, and consequently from equation (6)

$$p(r)=\frac{G\rho_0(M_0+\pi\rho_0)}{4r^4}+\pi\rho_0^2G\frac{\ln r}{r}+p_0.$$

If the functions $\varepsilon(\rho,T)$ and $\varkappa(\rho,T)$ are given explicitly, we can continue the integration and obtain the solutions to system (6)-(9). As can be seen from equations (19) and (20), the particular solution, obtained by the separation of variables, is $\varepsilon\sim\rho^{\lambda_1}T^{\lambda_2}$, $\varkappa\sim\rho^{\mu_1}T^{\mu_2}$; this is the form commonly used in astrophysics (Schwarzschild 1958).

The application of the above procedure to $X_2$ produces the following formulae:

$$M(r)=M_0r^3\qquad\rho(r)=3M_0/4\pi$$
$$p(r)=p_0-\frac{3GM_0^2}{8\pi}r^2.$$

We shall now briefly present the basis operators for some particular cases that are important from the physical point of view.

(i) Photon gas, $p\sim T^4$

$$X_1=-\tfrac{1}{3}r\frac{\partial}{\partial r}+\rho\frac{\partial}{\partial\rho}+\tfrac{1}{3}T\frac{\partial}{\partial T}$$
$$X_2=\tfrac{1}{3}r\frac{\partial}{\partial r}+M\frac{\partial}{\partial M}+\tfrac{1}{6}T\frac{\partial}{\partial T}$$
$$X_3=L\frac{\partial}{\partial L}$$
$$X_4=0.$$

(ii) Ideal gas, $p\sim\rho T$

$$X_1=-\tfrac{1}{3}r\frac{\partial}{\partial r}+\rho\frac{\partial}{\partial\rho}+\tfrac{1}{3}T\frac{\partial}{\partial T}$$
$$X_2=\tfrac{1}{3}r\frac{\partial}{\partial r}+M\frac{\partial}{\partial M}+\tfrac{2}{3}T\frac{\partial}{\partial T}$$
$$X_3=L\frac{\partial}{\partial L}$$

$$X_4 = 0.$$

(iii) Degenerate gas, $p \sim \rho^{5/3}$

$$X_1 = -\tfrac{1}{6} r \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial g} + \tfrac{1}{2} M \frac{\partial}{\partial M}$$

$$X_2 = 0$$

$$X_3 = L \frac{\partial}{\partial L}$$

$$X_4 = T \frac{\partial}{\partial T}.$$

The solution associated with $X_1$ is

$$\rho(r) = \rho_0 r^{-6} \qquad M(r) = M_0 - \frac{4\pi\rho_0}{3r^3} \qquad p(r) = p_0 + \frac{GM_0\rho_0}{7r^7} - \frac{4\pi G\rho_0^2}{10r^{10}}.$$

(iv) Relativistic degenerate electron gas, $p \sim \rho^{4/3}$

$$X_1 = -\tfrac{1}{3} r \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho}$$

$$X_2 = L \frac{\partial}{\partial L}$$

$$X_3 = 0$$

$$X_4 = T \frac{\partial}{\partial T}.$$

(v) Ideal and photon gas

$$X_1 = -\tfrac{1}{3} r \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} + \tfrac{1}{3} T \frac{\partial}{\partial T}$$

$$X_2 = 0$$

$$X_3 = L \frac{\partial}{\partial L}$$

$$X_4 = 0.$$

It is interesting to notice that the Eddington mass-luminosity dependence is not satisfied any longer, as there are no non-trivial invariants associated with $M$.

(vi) Ideal and degenerate gas

$$X_1 = -\tfrac{1}{6} r \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} + \tfrac{1}{2} M \frac{\partial}{\partial M} + \tfrac{2}{3} T \frac{\partial}{\partial T}$$

$$X_2 = 0$$

$$X_3 = L \frac{\partial}{\partial L}$$

$$X_4 = 0.$$

## 5. Conclusion

We have characterised, by computing infinitesimal operators, the structure of the group admissible by a system of equations describing Newtonian static stars in radiative equilibrium. We have shown that, in the most general case, the equations admit an infinite-parameter group of quasi-homologous transformations. These symmetries enforce appropriate equations of state. In the particular case of a five-parameter homologous group, the Stromgren results are recovered.

The equation of state (33) is very general. It contains both physical and non-physical situations.

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